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# On the influence of boundary conditions, Poisson's ratio and material non-linearity on the optimal shape

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**Abstract**

Design of a boundary in a material and in a structure is in principle an identical problem. However, for structures, we mostly assume the external loads to be given, while in the design for material moduli, the loads are evaluated from forced displacements and the loads are therefore design dependent. In the present study, we shall focus on the influence of these different load assumptions. Also, the influence of Poisson's ratio and of material non-linearity will be shown. Based on sensitivity analysis, we also obtain general information.

The objective may be the stiffest design, the strongest design or just a design of uniform energy density along the shape. In an energy formulation, it was proven earlier that these three objectives have the same solution, at least within the limits of geometrical constraints, including the parametrization. Without involving stress/strain fields, this proof holds good for 3D problems, for power-law non-linear elasticity and for anisotropic elasticity. With this background, the results of a number of influence studies are presented. 

*Keywords:* Optimal shape; Stiffness–strength; Material non-linearity; Anisotropy

## 1. Introduction

From the recent papers, Pedersen (1998a, 1999), we have the following results:

- The minimum compliance shape design (stiffest shape design) will have uniform energy density along the designed shape, as far as the geometrical constraints make this possible.
- If we furthermore assume that the highest energy densities are found at the designed shape, then the stiffest design will also be the strongest design, as defined by a design that minimizes the maximum energy density.

These rather general statements were backed up by a variety of solved problems, initiated around 1973. The aim of the statements was to get more mutual contact between three directions of active research. In mechanical and civil engineering, the focus has been on design for minimum stress concentration (Ding, 1986). In the material and more mathematically oriented research, the focus has been on design for

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minimum compliance (Vigdergauz, 1986). Some heuristic approaches have focused on design for uniform stress (Xie and Steven, 1997).

The assumptions behind the statements were naturally discussed with focus on the importance of geometrical constraints and parametrization. The results of a follow-up study have been published in Pedersen (1999) with specific interest in material design. The problem of designing the shape of a hole in order to maximize the homogenized bulk modulus was given special attention and a very simple parametrization was chosen. The same parametrization will be applied in the present paper. Agreement with earlier results of Vigdergauz (1986) and with Hashin–Shtrikman bounds (Grabovsky and Kohn, 1995), is obtained and the independence of Poisson's ratio on the optimal shapes was numerically found for these square-symmetric problems. As we shall see in the present paper this independence is generally not the case.

A further interesting result from this follow-up study is a very simple optimality criterion, which we shall here prove in more general terms. This optimality criterion gives rise to an efficient optimization procedure. Using this tool, the goal of this paper is to study the influence of the following aspects.

- influence from boundary conditions,
- influence from Poisson's ratio,
- influence from material non-linearity.

In summary, the main conclusions are

- The compliance of a structure or a material is rather insensitive to a variation of the shape.
- The energy density concentration is very sensitive to a variation of the shape.
- If a shape of constant energy density along the shape boundary is found, then the compliance is stationary.
- A design for stationary compliance may return high energy density concentration if geometrical constraints (parametrization) are a controlling parameter.
- In a finite domain problem, load applied by stress and load applied by forced displacements will give different optimal shapes.
- Especially for dense structures with given forced displacements, different Poisson's ratios of an isotropic material will in general give different optimal shapes. However, for given applied stresses the optimal shape is proven to be independent.
- Also, the non-linearity of a material will be reflected in the optimal design, but the influence is not very strong.

## 2. An optimality criterion and a method

From a more general point of view, we will here present the optimality criterion used in Pedersen (1999). This optimality criterion is the basis for the optimal designs to be shown. Let the shape $S$ (surface or curve) be described by the design parameters $h_i$. Then, the solid volume $V$ as well as the resulting elastic energy $U$ depend on these design parameters

$$S = S(h_i), \qquad V = V(h_i), \qquad U = U(h_i). \tag{2.1}$$

By variation from a given volume $V = \overline{V}$, we must satisfy

$$\mathrm{d}V = \sum_i \frac{\partial V}{\partial h_i} \mathrm{d}h_i = 0 \tag{2.2}$$

and, furthermore, we want to locate a solution of stationary elastic energy, i.e.

$$\mathrm{d}U = \sum_i \frac{\partial U}{\partial h_i} \mathrm{d}h_i = 0. \tag{2.3}$$

We shall first prove that a design with constant energy density along the shape boundary will be a solution and then later address the problem of determining such a design. Assume a non-linear elastic material, described by the potential of strain energy density $u = E\epsilon_{\mathrm{e}}^{q+1}/(q+1)$, where $E$ is a fixed modulus of elasticity and $\epsilon_{\mathrm{e}}$ is the reference strain (for 1D problems this gives $\sigma = E\epsilon^q$). Then the gradient of the total elastic energy $U$, with respect to any design parameter $h_i$ that do not change the actual load, is

$$\frac{\partial U}{\partial h_i} = -\frac{1}{q}\left(\frac{\partial U}{\partial h_i}\right)_{\text{fixed strain}}. \tag{2.4}$$

The proof of this can be found in Pedersen (1998b). This also holds for the involved forced displacements: from a theoretical point of view this gives design-dependent loads, but in a model with finite degrees of freedom and without direct element connections from the boundaries with forced displacements to the designed shape, this influence disappears.

Next, we assume a domain-divided model (say elements in a finite element model)

$$V = \sum_{\mathrm{e}} V_{\mathrm{e}}, \qquad U = \sum U_{\mathrm{e}} = \sum \overline{u}_{\mathrm{e}} V_{\mathrm{e}}, \tag{2.5}$$

where $\overline{u}_{\mathrm{e}}$ is the mean strain energy density in domain e. Then, the volume constraint (2.2) will be

$$\mathrm{d}V = \sum_i \left(\sum_{\mathrm{e}} \frac{\partial V_{\mathrm{e}}}{\partial h_i}\right) \mathrm{d}h_i = 0 \tag{2.6}$$

and the stationarity condition (2.3) with Eq. (2.4) is written

$$\mathrm{d}U = -\frac{1}{q}\sum_i \left(\sum_{\mathrm{e}} \overline{u}_{\mathrm{e}} \frac{\partial V_{\mathrm{e}}}{\partial h_i}\right) \mathrm{d}h_i = 0 \tag{2.7}$$

because $\overline{u}_{\mathrm{e}}$ is constant in a fixed strain field. It now follows that if $\overline{u}_{\mathrm{e}}$ is constant throughout the domains connected to the shape to be designed ($\partial V_{\mathrm{e}}/\partial h_i \neq 0$), then $\mathrm{d}V = 0$ implies $\mathrm{d}U = 0$. Note that, as shown in Pedersen (1999), we can have stationary solutions $\mathrm{d}U = 0$ without constant mean strain energy density, but as we see, not vice versa. However, without geometrical constraints (including the design parametrization), a stationary elastic energy will imply constant energy density along the boundary to be designed.

With a Lagrangian function $\mathscr{L} = U - \lambda\left(V - \overline{V}\right)$ to be made stationary, it follows from Eqs. (2.6) and (2.7) that the necessary optimality condition is

$$\sum_{\mathrm{e}} \overline{u}_{\mathrm{e}} \frac{\partial V_{\mathrm{e}}}{\partial h_i} \Bigg/ \sum_{\mathrm{e}} \frac{\partial V_{\mathrm{e}}}{\partial h_i} = \lambda \quad \text{for all } h_i. \tag{2.8}$$

The Lagrangian multiplier $\lambda$ is found from $\left(V = \overline{V}\right)$.

For a single inclusion hole as shown in Fig. 1 we only use three non-dimensional design parameters $\alpha, \beta$ and $\eta$. For this three-parameter problem the criterion (2.8) is simplified because the involved $\sum \partial V_{\mathrm{e}}/\partial h_i$ can be determined analytically. The resulting necessary optimality criterion is

$$\alpha \sum_s \overline{u}_s \frac{\partial V_s}{\partial \alpha} = \beta \sum_s \overline{u}_s \frac{\partial V_s}{\partial \beta} = \frac{\eta^2}{p(\eta)} \sum \overline{u}_s \frac{\partial V_s}{\partial \eta} = \lambda \tag{2.9}$$

restricted to summation over elements $s$ at the shape boundary and with $p(\eta)$ defined by

$$p(\eta) = 2\Psi(2/\eta) - \Psi(1/\eta) - \Psi((\eta+1)/\eta), \tag{2.10}$$

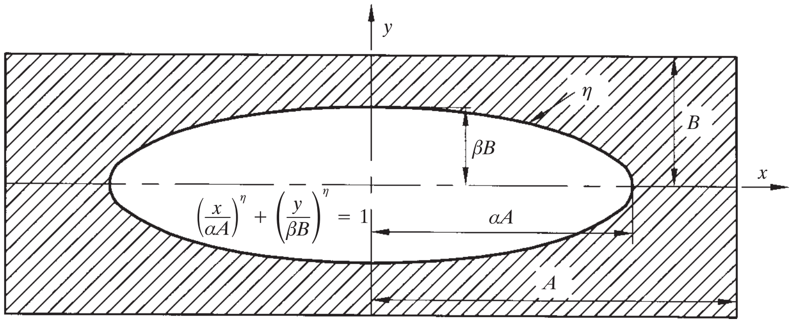


Fig. 1. The simplified shape parametrization used in Pedersen (1999) for material design.

where $\Psi$ is the Psi-function. For a detailed derivation of this result see Pedersen (1999). At first the $\alpha, \beta$ $\eta$-parametrization may seem too simple. However, the obtained results show almost uniform energy density along the optimal shapes and thus prove the versatility of this parametrization.

Optimal designs are located by an iterative method where the shape parameters are redefined with the goal of satisfying the optimality criterion (2.9). Volume $V$ depends monotonically on the parameters $\alpha, \beta$ and $\eta$ and rapid convergence is experienced. The procedure is a traditional optimality criterion method and details will therefore not be given.

Before the results from the study of influence of boundary conditions, Poisson's ratio and material non-linearity are presented; we shall show the influence from the volume constraint. With reference to Fig. 1, we define the relative density $\rho$ as the ratio of the area of the solid part (hatched) and the total area ($4AB$). The asymptotic case of $\rho \to 1$ is a small hole for which classical analytical design results are available. The other asymptotic case of $\rho \to 0$ gives a frame structure and we shall concentrate on the intermediate cases of say $0.2 \leqslant \rho \leqslant 0.8$.

The most simple case of design for maximum bulk modulus for a 2D material is with the parametrization in Fig. 1 described by a single parameter $\eta$, with $A = B$ and $\alpha = \beta$ uniquely determined by $\rho$ and $\eta$. The resulting optimal designs are shown in Fig. 2(a) with the curves in Fig. 2(b) giving the resulting Hashin–Shtrikman bounds. For this case the obtained optimal designs are independent of Poisson's ratio, but the resulting bulk moduli are highly dependent. That this is not the case in general will be shown in Section 4.

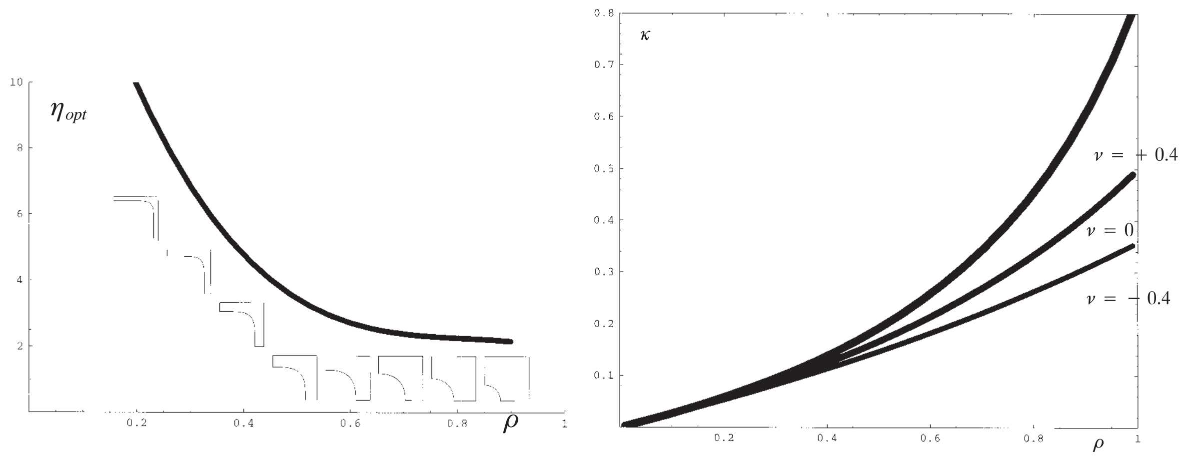


Fig. 2. (a) The optimal shape parameter $\eta$ as a function of relative volume density $\rho$ and (b) the resulting *2D-bulk* modulus $\kappa$ for three different values of Poisson's ratio $\nu$.

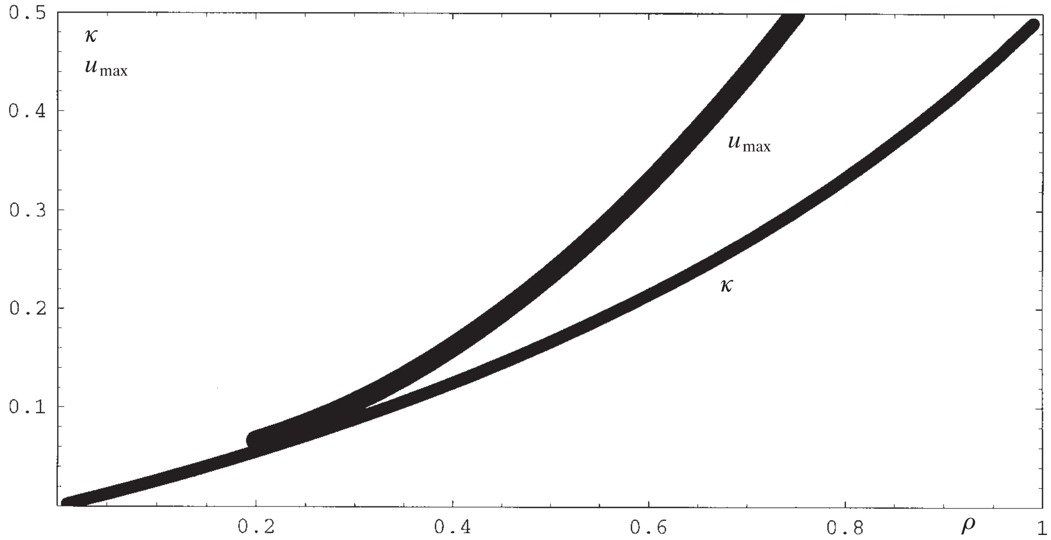


Fig. 3. Resulting *2D-bulk* modulus (equal to the Hashin–Shtrikman upper bound) and the resulting maximum energy density, both for the shapes in Fig. 1 and for zero Poisson's ratio.

Note in Fig. 2(b) that for negative Poisson's ratio an almost linear dependence of $\kappa$ as a function of $\rho$, while for positive Poisson's ratio a higher order dependence is observed. In a recent paper of Bendsøe and Sigmund (1999), this is discussed in detail.

In addition to this dependence of stiffness (bulk modulus) we show in Fig. 3 for the case of zero Poisson's ratio the dependence of strength (maximum energy density). We note that for low volume densities, say $\rho < 0.3$, we have moderate concentration of energy densities and almost linear dependence on $\rho$. However, for more continuum models, the strength (energy concentration) is more sensitive to $\rho$ than the stiffness (bulk modulus). This aspect relates to the recent work on topology optimization with stress constants, as by Duysinx and Bendsøe (1998).

## 3. Influence of boundary conditions

Without involving optimization it is useful to study solutions, where the shape of the model in Fig. 1 is determined only by the parameter $\eta$ (power of the super-elliptic function). We assume a symmetric model $A = B$, which with a constant solid area gives the values $\alpha = \beta$ for a chosen parameter $\eta$. Three different load cases are analyzed:

- forced *boundary displacements* at $x = A$, $y = B$, corresponding to pure mean dilatation

  $\overline{\epsilon}_y = \overline{\epsilon}_x = 0.001$,

- forced *boundary stresses* at $x = A$, $y = B$, corresponding to pure mean hydrostatic pressure

  $\sigma_x = \sigma_y = 0.0006 \quad$ (modulus of elasticity $C_{1111} = 1$),

- a *mixed problem* with forced boundary displacement at $x = A$, corresponding to $\overline{\epsilon}_x = 0.001$, and forced boundary stresses at $y = B$, corresponding to $\sigma_y = 0.0006$.

Note that for an infinitesimally small hole, these three problems are identical. With zero Poisson's ratio the Hashin–Shtrikman bound for the bulk modulus with relative solid area $\rho = 0.75$ is 0.3, and thus $\overline{\sigma}/\overline{\epsilon} = \sigma_x/(2\epsilon_x) = 0.3$ gives $\sigma_x = 0.6\epsilon_x$ (Pedersen, 1999). The results of the analyses are shown in Fig. 4 in

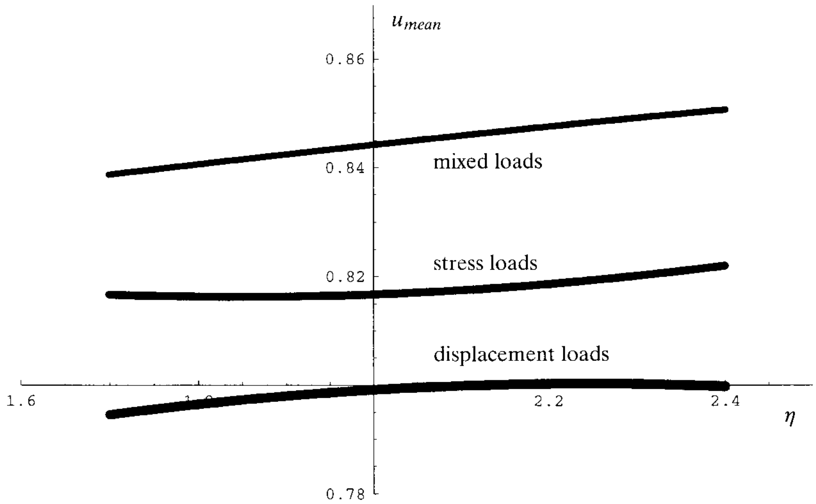


Fig. 4. Results of a one-parameter study, with mean strain energy density $u_{\text{mean}}(\eta)$ as a function of the shape parameter $\eta$. The value from the Hashin–Shtrikman bound is $u_{\text{mean}} = 0.8$.

terms of the mean strain energy densities in the solid areas (macroscopic bulk modulus $= (1/2)u_{\text{mean}} \cdot \rho = (3/8)u_{\text{mean}}$). Note, how insensitive the compliance (proportional to $u_{\text{mean}}$) is in this large design domain $1.7 < \eta < 2.4$.

The lower curve refers to the solutions with forced boundary displacements, and we notice a *maximum* of $u_{\text{mean}}$ at $\eta \approx 2.26$ and an agreement with the Hashin–Shtrikman bound.

The middle curve refers to the solutions with forced boundary stresses and we notice a *minimum* of $u_{\text{mean}}$ at $\eta \approx 1.84$ and that the bound is not reached. The upper curve represents the solutions with *mixed loads* and here no stationary solution is located. This is due to the fact that the stationary solution is not symmetric, i.e. $\alpha \neq \beta$.

Based on the optimality criterion (2.9) we then optimize the model with the mixed load case. The convergence history is shown in Fig. 5 and we notice an oscillating convergence of all the parameters $\eta$, $\alpha$ and $\beta$ as well as of the objective $u_{\text{mean}}$ for which a *stationary* solution is then found.

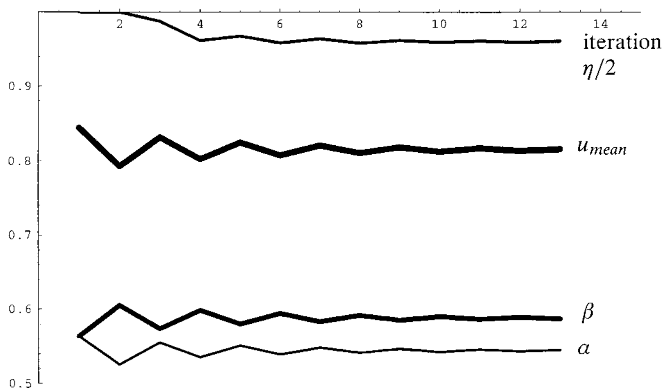


Fig. 5. Convergence history for the model with a mixed load case and three design parameters $\eta$, $\alpha$ and $\beta$. Also the objective $u_{\text{mean}}$ is shown.

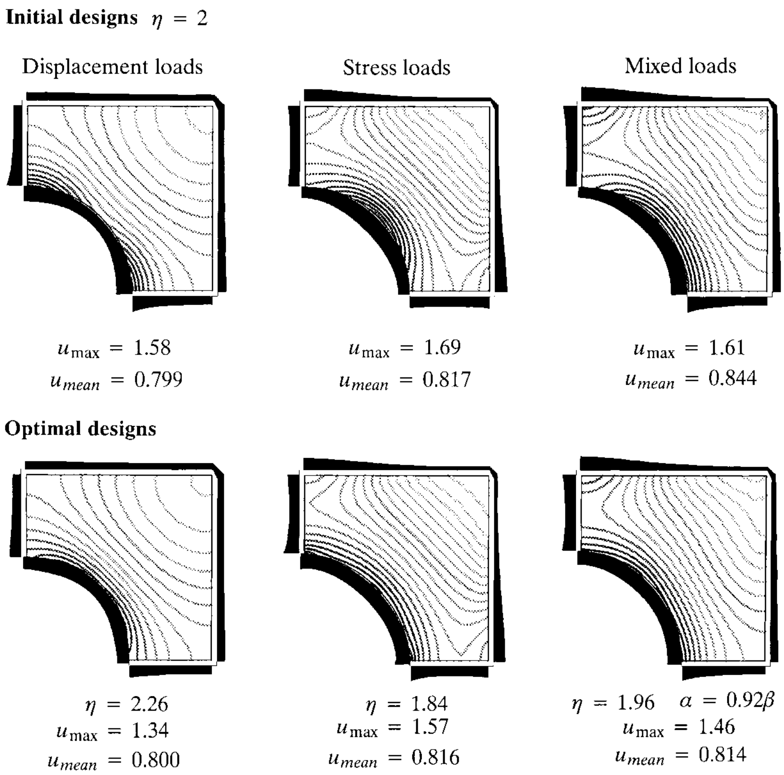


Fig. 6. Starting designs in the upper row with $\eta = 2$ and optimal solutions in the lower row with the left column: showing forced displacements with $\eta = 2.26$, middle column: showing forced stresses with $\eta = 1.84$, right column: showing mixed loads with $\eta = 1.96$ and $\alpha = 0.92\beta$. Energy densities along the boundaries in black and isolines for the larger principal stresses.

Lastly, in Fig. 6, we illustrate the different designs and the resulting boundary strain energy densities. In the initial non-optimal models, we see clearly the non-uniform distribution of energy density along the boundary of the hole. As seen, the corresponding optimal models (stationary total energy) have much more uniform distributions, although the simple parametrization does not allow for completely uniform solutions.

Even in recent literature on elastic compliances, as shown in the interesting paper by Zheng and Hwang (1997), the derivations are based on the given remote stresses. The present results show that the case of the given remote displacements is different, as we shall also see in Section 4.

## 4. Influence of Poissons ratio

The study in Section 3 was based on isotropic, zero Poisson's ratio linear elasticity. This was done to put the focus on the different boundary conditions. In earlier optimizations, it was found that the optimal shape depended strongly on the anisotropy (Pedersen et al., 1992), but a weak dependence on Poisson's ratio and on the non-linearity was found. We will in this section go deeper into the influence of Poisson's ratio and focus on the influence on the optimal shape design, rather than on the resulting stresses, strains and energies. It will be seen that for non-symmetric problems ($\bar{\epsilon}_x \neq \bar{\epsilon}_y$) the influence cannot always be neglected.

In the three papers by Thorpe and Jasiuk (1992), Cherkaev et al. (1992) and by Christensen (1993) the homogenized modulus of elasticity is proven to be independent of Poisson's ratio $v$ for the basic material of an isotropic composite. Note the critical comments in Zheng and Hwang (1997). In papers by Vigdergauz, as referred in Grabovsky and Kohn (1995), the shape for maximum bulk modulus is also found independent of $v$ although the bulk modulus itself, as shown in Fig. 2, depends strongly on $v$.

For more general problems with given forced displacements the objective and also the optimal shape depend on $v$, and the intention of this section is to show this by an example. Before doing this, we shall through sensitivity analysis get more detailed information and separate this influence of $v$ into the local influence when the stress or strain field is kept unchanged and the more global influence from a redistribution of these fields. From this it follows that the three sources are

- a general scaling factor,
- a further local change for fixed strains or fixed stresses,
- a global redistribution of displacements, strains and stresses.

For an *isotropic material*, the energy densities can be evaluated from the principal stresses $\sigma_{\mathrm{I}}$, $\sigma_{\mathrm{II}}$ and/or the principal strains $\epsilon_{\mathrm{I}}, \epsilon_{\mathrm{II}}$. Thus, we only need part of the constitutive relations

$$\left\{ \begin{matrix} \sigma_{\mathrm{I}} \\ \sigma_{\mathrm{II}} \end{matrix} \right\} = \begin{bmatrix} C_{1111} & C_{1122} \\ C_{1122} & C_{1111} \end{bmatrix} \left\{ \begin{matrix} \epsilon_{\mathrm{I}} \\ \epsilon_{\mathrm{II}} \end{matrix} \right\}. \tag{4.1}$$

With the same $C_{1111}$ all over the structure, this will only be a scaling factor and the ratio

$$\mu := C_{1122}/C_{1111} \quad -1 < \mu < 1 \tag{4.2}$$

is thus the important parameter. With a plane stress assumption we have, $\mu = v$ (Poisson's ratio) and with a plane strain assumption we have $\mu = v/(1-v)$. Also the ratio of principal stresses and the ratio of principal strains are important parameters and we define

$$\begin{aligned} \gamma &:= \epsilon_{\mathrm{II}}/\epsilon_{\mathrm{I}} \quad -1 \leqslant \gamma \leqslant 1, \\ \zeta &:= \sigma_{\mathrm{II}}/\sigma_{\mathrm{I}} \quad -1 \leqslant \zeta \leqslant 1. \end{aligned} \tag{4.3}$$

Then, the constitutive behaviour is rewritten to

$$\sigma_{\mathrm{I}} \left\{ \begin{matrix} 1 \\ \zeta \end{matrix} \right\} = C_{1111}\epsilon_{\mathrm{I}} \begin{bmatrix} 1 & \mu \\ \mu & 1 \end{bmatrix} \left\{ \begin{matrix} 1 \\ \gamma \end{matrix} \right\}. \tag{4.4}$$

It follows from Eq. (4.4) that we can express $\zeta$ in $\mu$ and $\gamma$ or $\gamma$ in $\mu$ and $\zeta$

$$\zeta = (\mu + \gamma)/(1 + \mu\gamma); \qquad \gamma = (\zeta - \mu)/(1 - \mu\zeta), \tag{4.5}$$

which is always valid as $\mu\gamma > -1$ and $\mu\zeta < 1$. The conditions in Eq. (4.3) are satisfied, i.e., the numerically larger $\epsilon_{\mathrm{I}}$ and $\sigma_{\mathrm{I}}$ are aligned. Note that $\gamma = 0$ implies $\zeta = \mu$ and $\zeta = 0$ implies $\gamma = -\mu$.

For linear elasticity, the strain energy density $u$ and the stress energy density $u^C$ have the same value, which is determined by

$$u = u^C = \tfrac{1}{2}(\sigma_{\mathrm{I}}\epsilon_{\mathrm{I}} + \sigma_{\mathrm{II}}\epsilon_{\mathrm{II}}) = \tfrac{1}{2}\sigma_{\mathrm{I}}\epsilon_{\mathrm{I}}(1 + \zeta\gamma). \tag{4.6}$$

Using relations (4.4) and (4.5), we can express this as

$$u = \tfrac{1}{2}C_{1111}\epsilon_{\mathrm{I}}^2\left(1 + 2\mu\gamma + \gamma^2\right) \tag{4.7}$$

or by

$$u^C = \frac{\sigma_{\mathrm{I}}^2}{2C_{1111}(1-\mu^2)}\left(1 - 2\mu\zeta + \zeta^2\right). \tag{4.8}$$

We have the *global scaling factors* $C_{1111}$ and $(1-\mu^2)$. In fact, for the plane stress models, we have $C_{1111}(1-\mu^2) = (E/(1-\nu^2))(1-\nu^2) = E$ and therefore no scaling. The main conclusion from Eqs. (4.7) and (4.8) is that even in a fixed strain field or a fixed stress field the energy density will, in addition to the global scaling factor, depend on the ratio $\mu = C_{1122}/C_{2222}$ and we will only have no influence with unidirectional strain ($\gamma = 0$) or with unidirectional stress ($\zeta = 0$).

At the free boundary to be designed, we have unidirectional stress. Thus, if we can prove the stress field to be unchanged, then the optimal shape will be independent of $\mu$, i.e. independent of Poisson's ratio. With the given boundary stresses in equilibrium this is true for 2D linear elastic problems (Muskhelishvili, 1963, pp. 160–161). Numerical results for our actual problems confirm this.

However, with given boundary displacements the optimal design will depend on $\mu$, i.e. depend on Poisson's ratio. To get some information for this case also, we shall then set up a global sensitivity analysis, primarily with respect to the nodal displacements in a finite element model. In addition to the ratio $\mu = C_{1122}/C_{1111}$ we have for isotropic models $C_{1212}/C_{1111} = (1-\mu)/2$. With this being the case for the total model, the total stiffness matrix is linear in $\mu$ and we have

$$[S]\{D\} = ([S_0] + \mu[S_1])\{D\} = \{A\}, \tag{4.9}$$

where $\{D\}$ are the nodal displacements and $\{A\}$ the nodal load obtained by either prescribed boundary stresses or by prescribed boundary displacements. For $\{A\}$ not directly dependent on $\mu$ we must solve

$$[S]\frac{\partial\{D\}}{\partial\mu} = -[S_1]\{D\} \tag{4.10}$$

and for $\{A\} = -([S_0] + \mu[S_1])\{D_f\}$, where $\{D\}_f$ is the vector of forced displacements, we must solve

$$[S]\frac{\partial\{D\}}{\partial\mu} = -[S_1]\Big(\{D_f\} + \{D\}\Big) \tag{4.11}$$

to get $\partial\{D\}/\partial\mu$ and from this the sensitivities of the strain and the stress fields.

For the symmetric problems in Pedersen (1999), we have seen independence on $\mu$, but for non-symmetric problems this is not the case. In Fig. 7, we show the rather strong influence for an actual problem with only weak non-symmetry described by $\bar{\epsilon}_x = 1.2\bar{\epsilon}_y$.

## 5. Influence of elastic non-linearity

In the sensitivity analysis relative to Poisson's ratio, we have identified three sources of possible influence from the ratio $\mu = C_{1122}/C_{1111}$. On the local level, the scaling influence through $C_{1111}$ and $(1-\mu^2)$ and the further influence depending on the ratio of principal stresses or strains according to Eqs. (4.7) and (4.8). On the global level, we have the influence through a redistribution of the stress/strain fields. We shall perform a somewhat similar analysis for non-linear elasticity described by $\sigma_e = E\epsilon_e^q$, where $\sigma_e, \epsilon_e$ are the effective stress, strain defined relative to energy densities (Pedersen, 1998b). In general, we have

$$u + u^C = \sigma_I\epsilon_I + \sigma_{II}\epsilon_{II} \tag{5.1}$$

and, as also shown in Pedersen (1998b),

$$u^C = qu. \tag{5.2}$$

It then follows that the generalizations of Eqs. (4.7) and (4.8) are

$$u = \frac{1}{1+q}C_{1111}\epsilon_I^2\left(1 + 2\mu\gamma + \gamma^2\right) \tag{5.3}$$

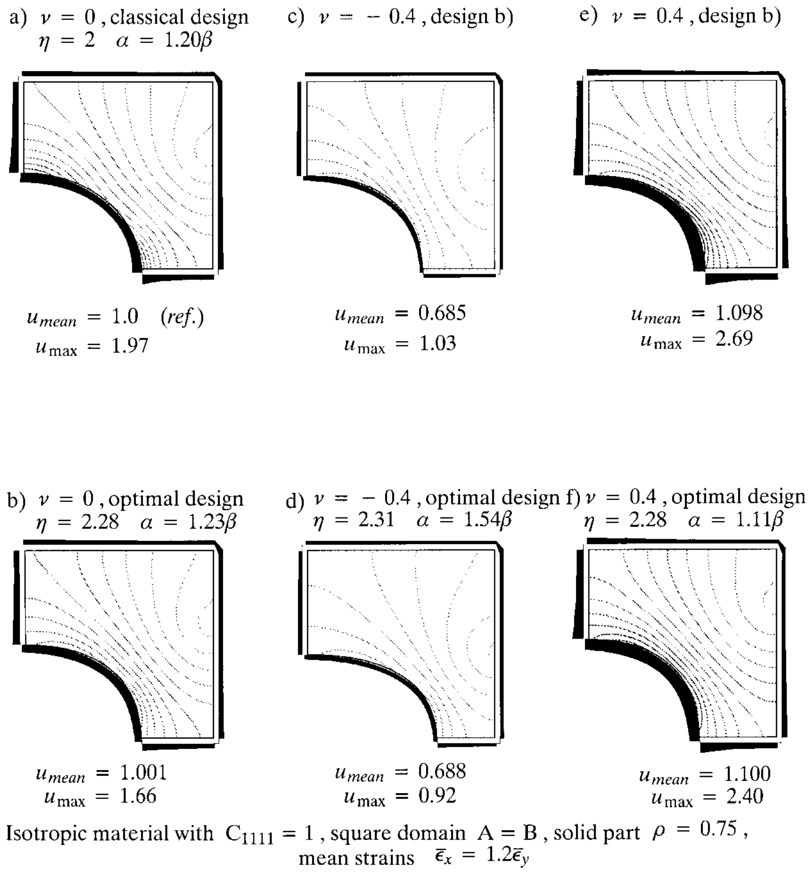


Fig. 7. Influence of Poisson's ratio for an isotropic material, non-symmetric displaced $\bar{\epsilon}_x = 1.2\bar{\epsilon}_y$. Strain energy densities at boundaries in black and isolines for the larger principal stresses.

and

$$u^C = \frac{q}{1+q}\frac{\sigma_{\rm I}^2}{C_{1111}(1-\mu^2)}\left(1 - 2\mu\zeta + \zeta^2\right). \tag{5.4}$$

On the local level, we have directly from Eqs. (5.3) and (5.4) that in a fixed strain field or in a fixed stress field we get *proportional changes* with respect to the non-linearity parameter $q$

$$\left(\frac{\partial u}{\partial q}\right)_{\text{fixed strains}} = -\frac{1}{1+q}u, \tag{5.5}$$

$$\left(\frac{\partial u^C}{\partial q}\right)_{\text{fixed stresses}} = \frac{1}{q(1+q)}u^C. \tag{5.6}$$

Thus, this local influence from the non-linearity parameter $q$ will not change a characteristic of constant energy density and the optimality criterion will still be satisfied.

We then perform a sensitivity analysis to get information about the global change of the displacement field $\mathrm{d}\{D\}/\mathrm{d}q$, strain field and stress field. The finite element formulation for this is

$$[S]\frac{d\{D\}}{dq} = \frac{d\{A\}}{dq} - \frac{d[S]}{dq}\{D\}. \tag{5.7}$$

Let us assume a constant stress, strain, energy density element and that the strain state in element e is of magnitude $\epsilon_e$ that relates to the actual strain energy density $u_e$ by

$$u_e = \frac{1}{q+1}E\epsilon_e^{q+1}, \tag{5.8}$$

where $E$ is a fixed modulus of elasticity, see Pedersen (1998b) for discussion of this definition of $\epsilon_e$. Then the element stiffness matrix $[S_e]$ can be written with relative strain $\tilde{\epsilon}_e = \epsilon_e/\epsilon_o$

$$[S_e] = \tilde{\epsilon}_e^{q-1}\left[\tilde{S}_e\right], \tag{5.9}$$

where $\left[\tilde{S}_e\right]$ is not dependent on $q$. Then, we get

$$\frac{d[S_e]}{dq} = \ln\left(\tilde{\epsilon}_e\right)\epsilon_e^{q-1}\left[\tilde{S}_e\right] = \ln\left(\tilde{\epsilon}_e\right)[S_e]. \tag{5.10}$$

Thus, for elements of equal energy density (equal effective strains $\epsilon_e$), we get proportional change of the stiffness matrices. From this follows the result in Pedersen and Taylor (1993) that in thickness design the optimal design does not depend on the non-linearity $q$.

However, for shape design the effective strain $\epsilon_e$ is only constant along the boundary and thus a redistribution of displacements, strains and stresses will take place. As we see in the example in Fig. 8, the

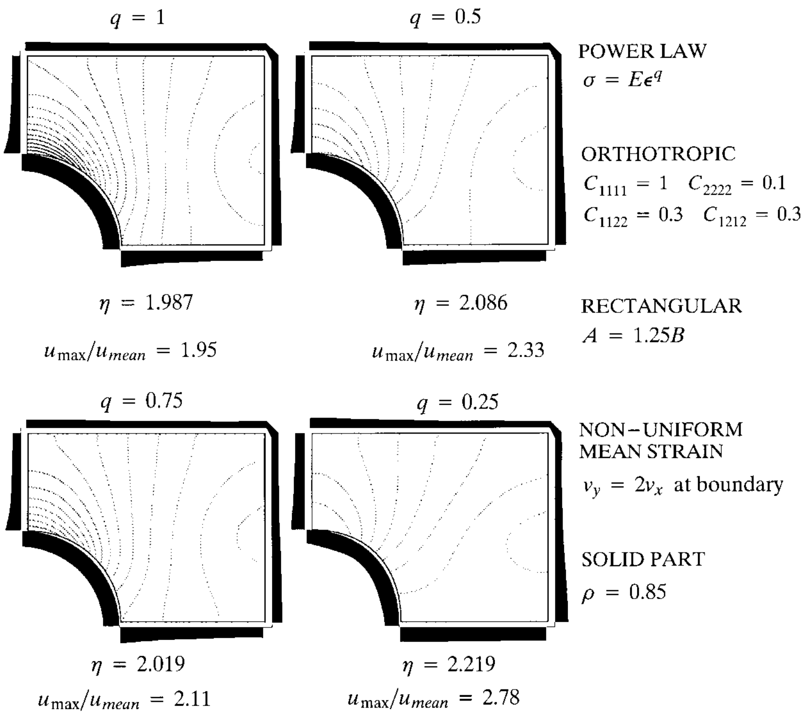


Fig. 8. Change in optimal design for increasing non-linearity $q = 0.75, 0.50, 0.25$. Energy densities at boundaries in black and isolines for the larger principal stresses.

optimal shape only changes slightly, probably due to the fact that the change in $\epsilon_e$ is only in the direction orthogonal to the shape.

## 6. Conclusion

The examples in this follow-up study confirm the finding that compliance is very insensitive to changes in the boundary shape of a hole. In contrast to this, the concentration of energy density is very sensitive and we need to be careful with the details of the shape design.

A simple optimality criterion method was used to optimize shapes for given boundary displacements as well as with given boundary stresses. Not only the convergence history but also the optimal design depends on these different boundary conditions. An example, which for an infinite domain should give the same results, shows this aspect for a finite model.

In the optimal design for maximum bulk modulus (symmetric problem) there is no influence on the optimal design from Poisson's ratio. In a sensitivity analysis, the dependence is discussed more generally and an example shows the rather strong dependence for a non-symmetric problem with forced boundary displacements. It follows from the sensitivity analysis that this dependence is related to the overall redistribution of the displacement field when Poisson's ratio is changed. With given boundary stresses, the optimal shape is proven to be independent of Poisson's ratio for cases with unchanged stress fields, which include the test cases.

In a somewhat similar sensitivity analysis, the influence from material non-linearity is analyzed. Again, an overall redistribution of the displacement field causes a change in the optimal shape. However, the change in the optimal shape is not very drastic for the test cases, compared with the change in the level of energy densities.

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